Outcome Classes in Impartial Games

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Impartial games

- Both Left and Right have the same options in any game state.
- Examples: GEOGRAPHY, NIM.
- Is CONNECT-FOUR impartial?
- What about outcome classes $\mathcal{L}$ and $\mathcal{R}$?

Proof by induction, $\mathcal{L} \cup \mathcal{R} = \emptyset$

- All leaves are in $\mathcal{P}$.
- $G^L = G^R$
- Outcome table:

	Some $G^R \in \mathcal{R} \cup \mathcal{P}$	All $G^R \in \mathcal{L} \cup \mathcal{N}$
Some $G^L \in \mathcal{L} \cup \mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$
All $G^L \in \mathcal{R} \cup \mathcal{N}$	$\mathcal{R}$	$\mathcal{P}$

Partition Theorem

If we partition a finite game in mutually exlusive sets $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

1. every option of a position in $\boldsymbol{A}$ is in $\boldsymbol{B}$, and
2. every position in $\boldsymbol{B}$ has at least one option in $\boldsymbol{A}$,

then $\boldsymbol{A} \subset \mathcal{P}$ and $\boldsymbol{B} \subset \mathcal{N}$.

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Proof by mutual induction.

1. Assuming $\boldsymbol{B} \subset \mathcal{N}$, prove $\boldsymbol{a} \in \boldsymbol{A} \implies \boldsymbol{a} \in \mathcal{P}$.
2. Assuming $\boldsymbol{A} \subset \mathcal{P}$, prove $\boldsymbol{b} \in \boldsymbol{B} \implies \boldsymbol{b} \in \mathcal{N}$.
3. Find a base case and do induction (but on what?).

Partition Theorem conclusion

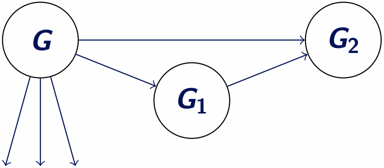
A position is

- a $\mathcal{P}$-position if all of its options are $\mathcal{N}$-positions, and
- an $\mathcal{N}$-position if at least one of its options is a $\mathcal{P}$-position.

Exercise: **3** $\times$ **3** Cram.

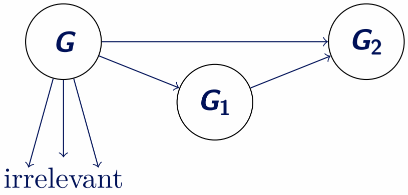
Bottleneck Principle

Let G be an impartial game with options G_1 and G_2 (it could have more options), and G_2 is the only option of G_1.



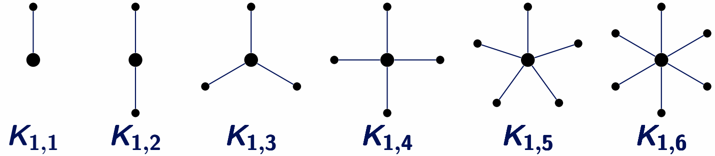
Bottleneck Principle

Let $\boldsymbol{G}$ be an impartial game with options $\boldsymbol{G_1}$ and $\boldsymbol{G_2}$ (it could have more options), and $\boldsymbol{G_2}$ is the only option of $\boldsymbol{G_1}$, then $\boldsymbol{G}$ is an $\mathcal{N}$-position.



Cutthroat Stars

In CUTTHROAT STARS you can remove a vertex and all adjacent edges. At least one edge must be removed this way. Or, *shrink* and *supernova*.



Subtraction

In SUBTRACTION(S) you have n counters, and can subtract any $s \in S$ (assuming enough counters are left), leaving $n - s$ counters.

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1. If G_n is a game of Subtraction$(\{1, 3, 4\})$ with n counters left, which G_n are in $\mathcal{P}$?

2. What about Subtraction$(\{2^n : n = 0, 1, 2, \ldots\})$?

Two more games

1. Greedy Nim is just like Nim, except you can only take from the largest heap (or any of them if there are multiple).
2. The Common Divisor game is played with multiple heaps, and at any point you may remove from one heap a common divisor of all heaps. Here $\gcd(0, n) = n$. Who wins on the two-heap game?