Seminar (Combinatorial) Algorithms

Universiteit Leiden

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Spring 2019, Snellius 408

Tuesday 5.2.2019, 11:00–13:00

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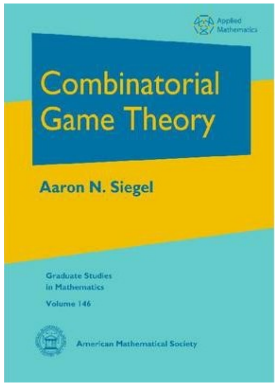
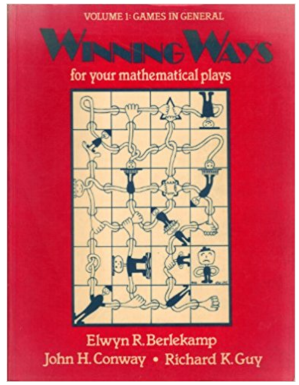
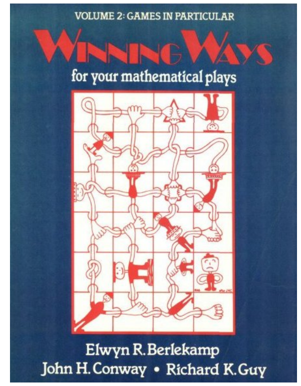
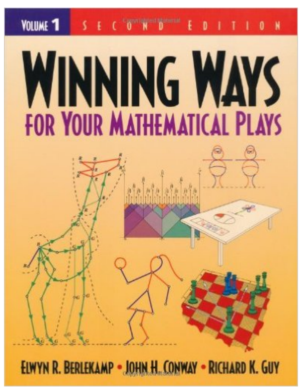
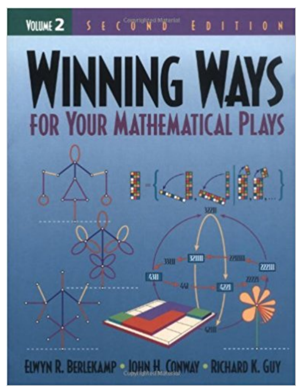
We discuss texts dealing with Combinatorial Games.

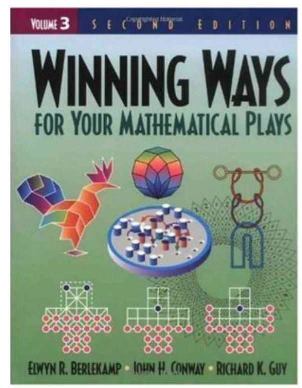


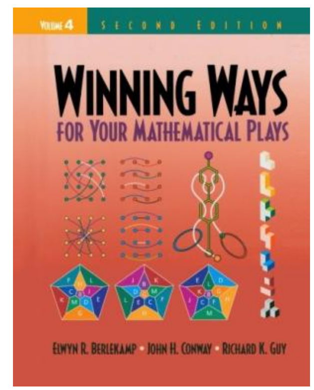












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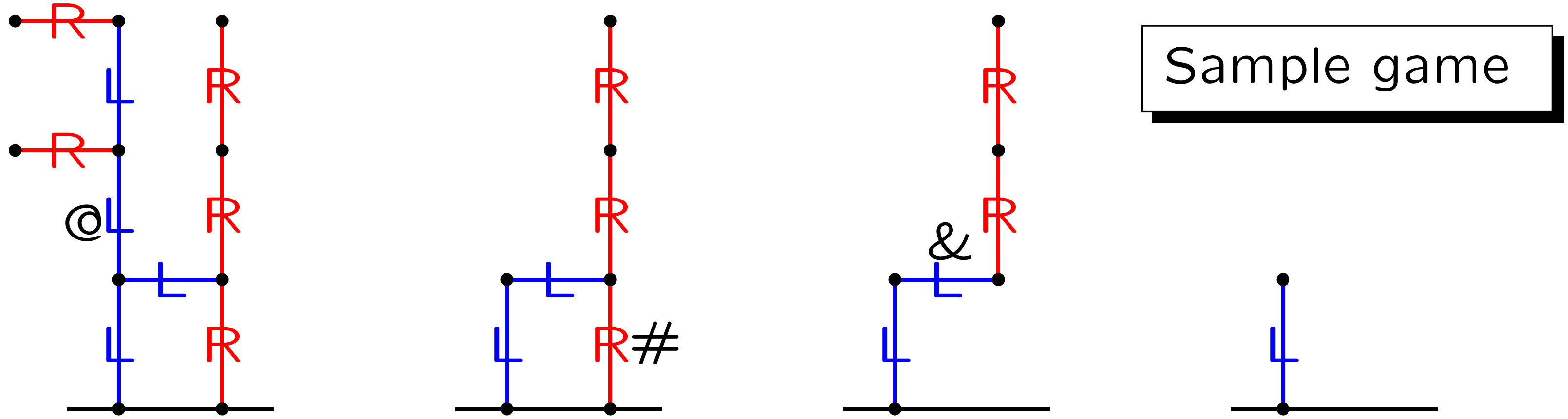
First we examine three example games:

- Hackenbush
- Nim
- Clobber

And then:

- Literature
- How is the seminar organized?

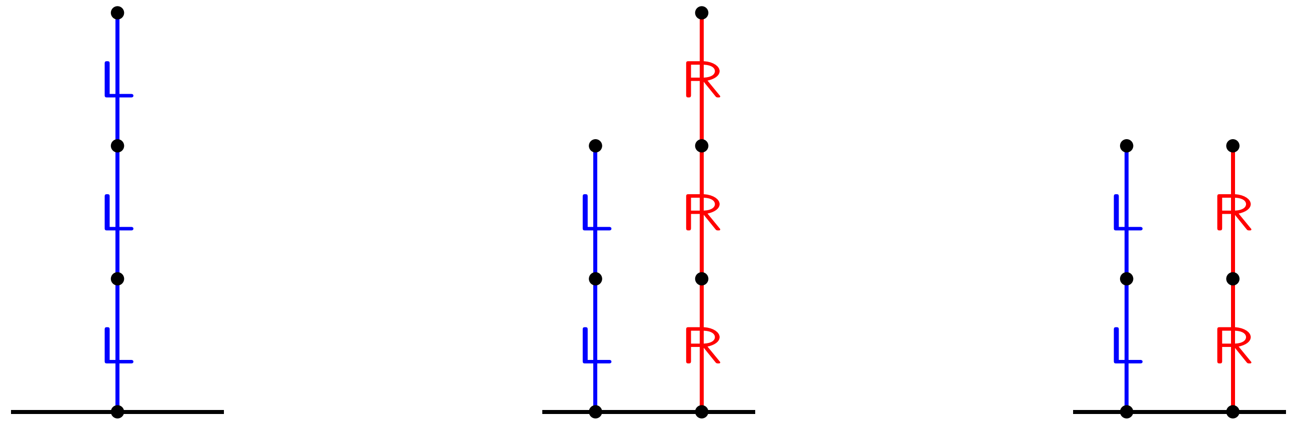
In the game (Blue-Red-)Hackenbush Left = she and Right = he alternately remove a bLue or a Red edge. All edges that are no longer connected to the ground, are also removed. *If you cannot move, you lose!*



Left chooses @, Right # (stupid), Left &. Now Left wins because Right cannot move.

By the way, Right can win here, whoever starts!

When playing Hackenbush, what is the value of a position?



value 3 value $2 - 3 = -1$ value $2 - 2 = 0$

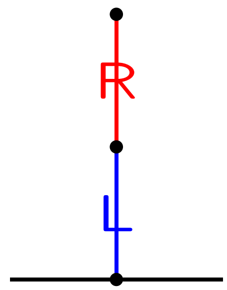
value > 0: Left wins (whoever starts) $\mathcal{L}$

value $= 0$: first player loses $\mathcal{P}$

value < 0: Right wins (whoever starts) $\mathcal{R}$

Remarkable(*): Hackenbush has no "first player wins"! $\mathcal{N}$

But what is the value of this position?



If Left begins, she wins immediately. If Right begins, Left can still move, and also wins. So Left always wins. Therefore, the value is > 0.

Is the value equal to 1?

If the value in the left hand side position would be 1, the value of the right hand side position would be $1+(-1)=0$, and the first player should lose. Is this true?

No! If Left begins, Left loses, and if Right begins Right can also win. So Right always wins (i.e., can always win), and therefore the right hand side position is < 0, and the left one is between 0 and 1.

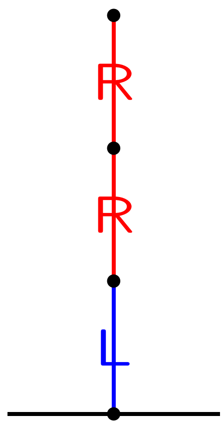
The left hand side position is denoted by $\{\, 0 \mid 1 \,\}$.

Note that the right hand side position does have value 0: the first player loses. And so we have:

$$\{\, 0 \mid 1 \,\} + \{\, 0 \mid 1 \,\} + (-1) = 0,$$

and “apparently” $\{\, 0 \mid 1 \,\} = 1/2$.

We denote the value of a position where Left can play to (values of) positions from the set L and Right can play to (values of) positions from the set R by $\{\, L \mid R \,\}$.

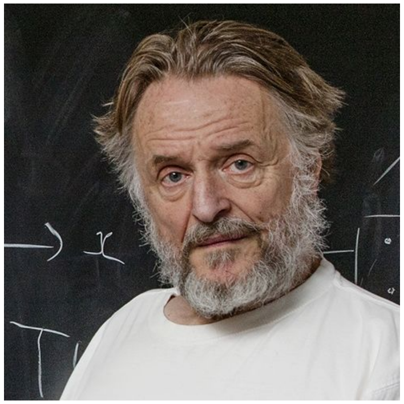


The value is $\{\, 0 \mid \frac{1}{2}, 1 \,\} = \frac{1}{4}$.

Simplicity rule:The value is always the "simplest" number between left and right set: the smallest integer — or the dyadic number with the lowest denominator (power of 2).

Give a position with value 3/8.

Show that $\{\, 0 \mid 100 \,\} = 1$.



Donald E.(Ervin) Knuth
1938, US
NP; KMP
$\TeX$
change-ringing; 3:16
The Art of Computer Programming

John H.(Horton) Conway
1937, UK $\to$ US
Co_1, Co_2, Co_3
Doomsday algoritme
game of Life; Angel problem
Winning Ways for your Mathematical Plays

Surreal numbers

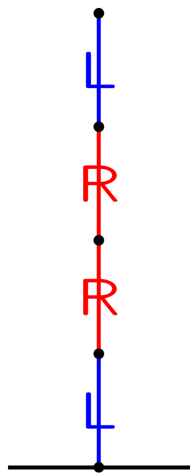
In this way we define surreal numbers: “decent” pairs of sets of previously defined surreal numbers: all elements from the left set are smaller than those from the right set.

Start with $0 = \{\, \emptyset \mid \emptyset \,\} = \{\, \text{nothing} \mid \text{nothing} \,\} = \{\, \mid \,\}$: the game where both Left and Right have no moves at all, and therefore the first player loses: born on day 0.

And then $1 = \{\, 0 \mid \,\}$ en $-1 = \{\, \mid 0 \,\}$, born on day 1.

And $42 = \{\, 41 \mid \,\}$, born on day 42.

And $\frac{3}{8} = \{\, \frac{1}{4} \mid \frac{1}{2} \,\}$, born on day 4.



Sets can be infinite: $\pi = \{\, 3, 3\frac{1}{8}, 3\frac{9}{64}, \ldots \mid 4, 3\frac{1}{2}, 3\frac{1}{4}, 3\frac{3}{16}, \ldots \,\}$.

We define, e.g.:

$$\varepsilon = \{\, 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \,\},$$

an "incredibly small number", and

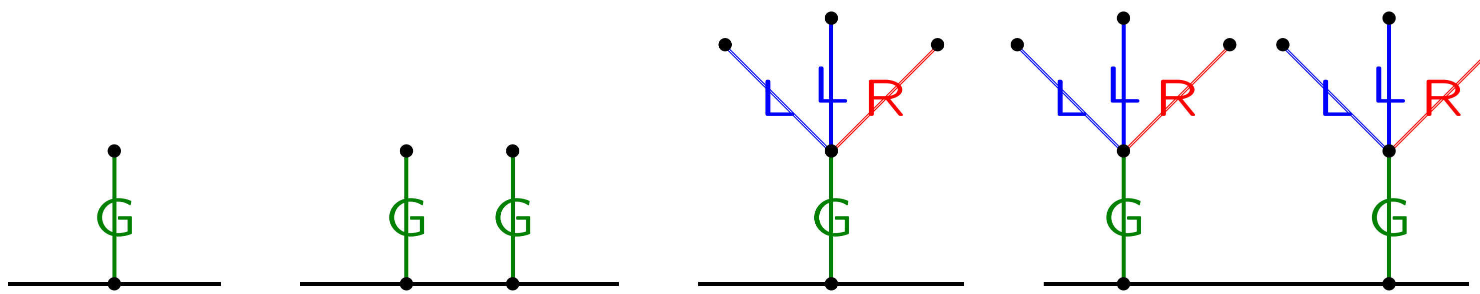
$$\omega = \{\, 0, 1, 2, 3, \ldots \mid \,\} = \{\, \mathbf{N} \mid \emptyset \,\},$$

a "terribly large number, some sort of ∞".

Then we have $\varepsilon \cdot \omega = 1$ — if you know how to multiply.

And then $\omega + 1$, $\sqrt{\omega}$, ω^{ω}, $\varepsilon/2$, and so on!

In Red-Green-Blue-Hackenbush we also have Green edges, that can be removed by both players.



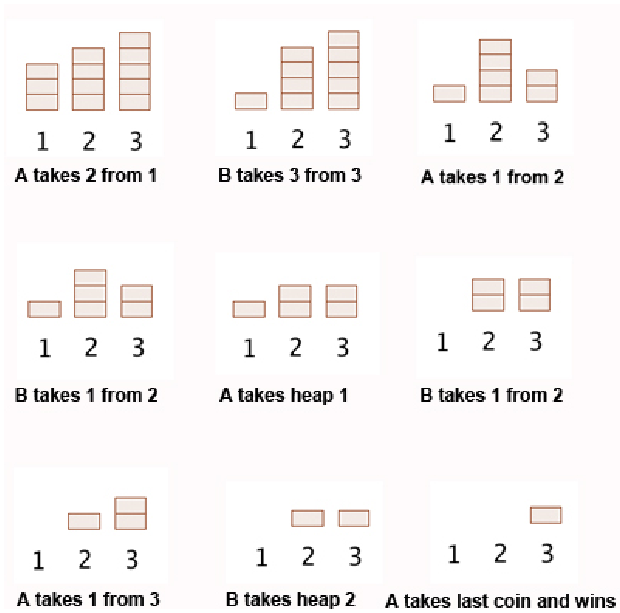
The first position has value $* = \{ 0 \mid 0 \}$ (not a surreal number), because the player to move can win.

The second position is $* + * = 0$ (player to move loses).

The third position is a first player win.

The fourth position is a win for Left (whoever begins), and is therefore > 0.

In the Nim game we have several stacks of tokens = coins = matches. A move consists of taking a nonzero number of tokens from one of the stacks. If you cannot move, you lose ("normal play").



The game is impartial: both players have the same moves. (And for the misère version: if you cannot move, you win.)

For Nim we have Bouton's analysis from 1901.

We define the nim-sum $x \oplus y$ of two positive integers x en y as the bitwise XOR of their binary representions: addition without carry. With two stacks of equal size the first player loses ($x \oplus x = 0$): use the "mirror strategy".

A nim game with stacks of sizes $a_1, a_2, \ldots, a_k$ is a first player loss exactly if $a_1 \oplus a_2 \oplus \ldots \oplus a_k = 0$. And this sum is the Sprague-Grundy value.

We denote a game with value m by $*m$ (the same as a stalk of m green Hackenbush edges; not a surreal number). And $*1 = *$. So if $m \neq 0$ the first player loses.

The Sprague-Grundy Theorem roughly says that every impartial gave is a Nim game.

With stacks of sizes 29, 21 and 11, we get $29 \oplus 21 \oplus 11 = 3$:

```
11101      29
10101      21
 1011      11
-----      --
00011       3
```

So a first player win, with unique winning move $11 \to 8$.

Why this move, and why is it unique?

How to add these "games" (we already did)? Like this:

$$a + b = \{\, A_L + b, a + B_L \mid A_R + b, a + B_R \,\}$$

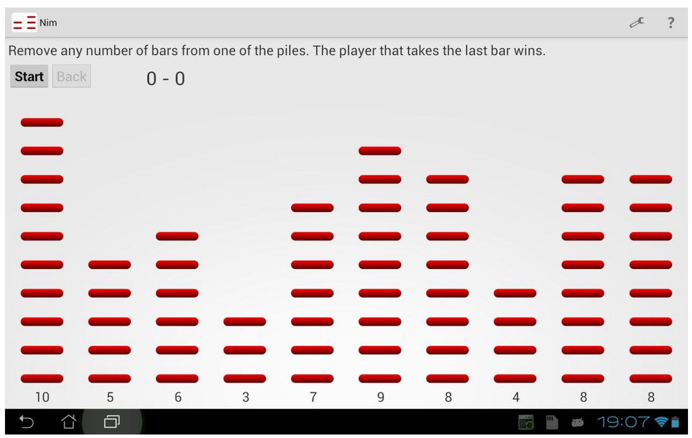
if $a = \{\, A_L \mid A_R \,\}$ and $b = \{\, B_L \mid B_R \,\}$.
Here we put $u + \emptyset = \emptyset$ and $u + V = \{u + v : v \in V\}$.

This corresponds with the following: you play two games in parallel, and in every move you must play in exactly one game: the disjunctive sum.

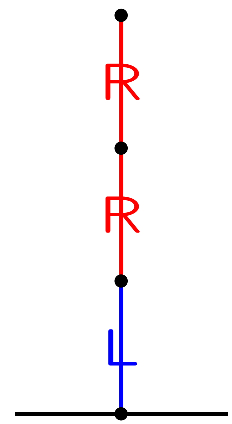
> Verify that $1 + \frac{1}{2} = \{\, 1 \mid 2 \,\} = \frac{3}{2}$.

See Claus Tøndering's paper

Now consider this addition of two game positions, with on the left a Nim position and on the right a Hackenbush position:



$+$

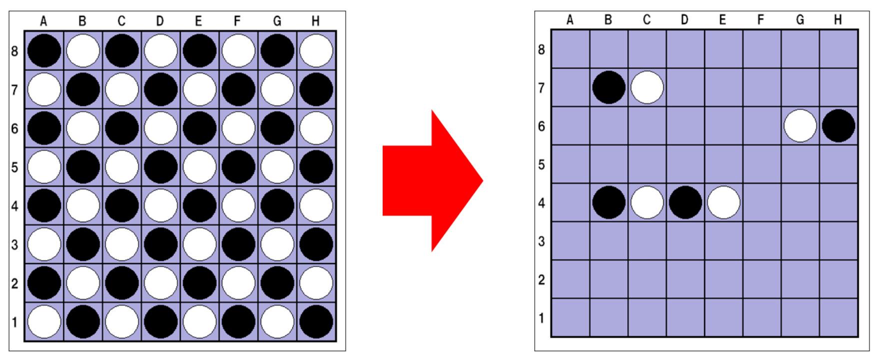


Then this sum is > 0, it is a win for Left! More general: $*m + 1/1024 > 0$.

We finally play Clobber, on an m times n board, with white (Right) and black (Left) stones. A stone can capture = "clobber" a directly adjacent stone from the other color. If you cannot move, you lose.

Some examples:

●○ $= \{ 0 \mid 0 \} = *$



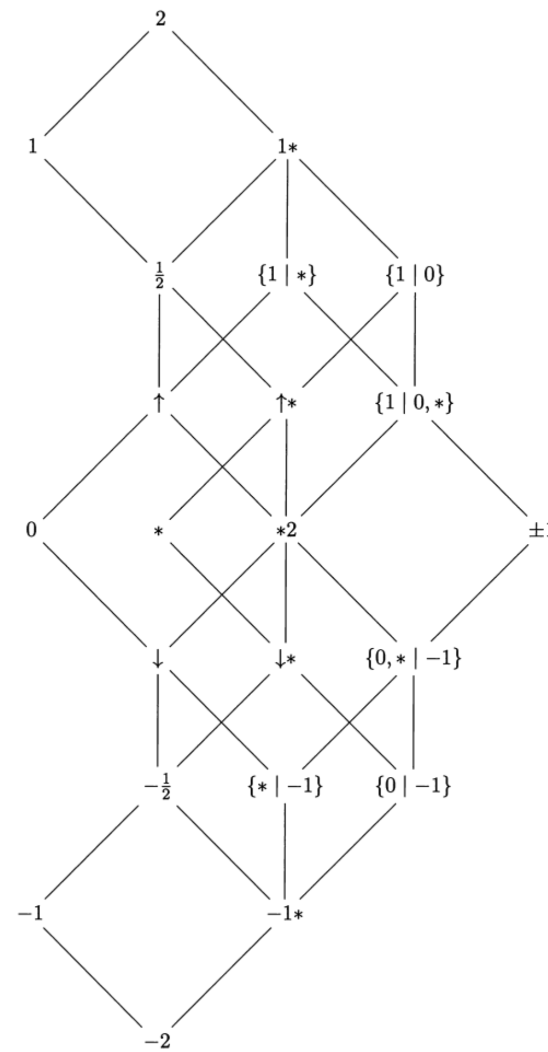
●●○ $= \{ 0 \mid * \} = \uparrow > 0$

●○●○●○ $= 0$

●○●○●○●○●○ $= \pm(\uparrow, \uparrow^{[2]} *, \{0 | \uparrow, \pm(*, \uparrow)\}, \{\Uparrow * | \downarrow, \pm(*, \uparrow)\})$

From the Siegel book:

$$0 < \{\, 0 \mid * \,\} = \uparrow < 1/2 < \{\, 1 \mid 1 \,\} = 1* < 2$$



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<tr><th colspan="2"></th><th>-1</th><th>$0, *$</th><th>0</th><th>$*$</th><th>1</th><th>—</th></tr>
<tr><th rowspan="6">Left Options</th><th>1</th><td>± 1</td><td>$\{1 \mid 0, *\}$</td><td>$\{1 \mid 0\}$</td><td>$\{1 \mid *\}$</td><td>$1*$</td><td>2</td></tr>
<tr><th>$0, *$</th><td>$\{0, * \mid -1\}$</td><td>$*2$</td><td>$\uparrow *$</td><td rowspan="2">$\uparrow$</td><td rowspan="2">$\frac{1}{2}$</td><td rowspan="2">1</td></tr>
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<tr><th>$*$</th><td>$\{* \mid -1\}$</td><td colspan="2">$\downarrow$</td><td colspan="3" rowspan="3">0</td></tr>
<tr><th>-1</th><td>$-1*$</td><td colspan="2">$-\frac{1}{2}$</td></tr>
<tr><th>—</th><td>-2</td><td colspan="2">-1</td></tr>
</table>

Two main references:

Siegel:
A.N. Siegel, Combinatorial Game Theory, AMS, 2013.

WinningWays:
E.R. Berlekamp, J.H. Conway and R.K. Guy, Winning Ways for your Mathematical Plays, 1982/2001.
(Note that there are two editions: the first has two volumes, the second has four volumes. Page numbers below refer to the second edition, and differ a little from those of the first edition. In all cases: volume 1.)

And the subjects (first half):

1. Introduction, Siegel, pp. 1–7.
2. Introduction (continued), Siegel, pp. 8–14.
3. Nim and Sprague-Grundy, Siegel, pp. 179–183;
   Wikipedia: en.wikipedia.org/wiki/Sprague_Grundy_theorem.
4. Heap games, Siegel, pp. 184–188.
5. Octal games, Siegel, pp. 188–192.
6. Ski-jumps, WinningWays, pp. 7–13.
7. Simplicity rule (intuition), WinningWays, pp. 19–28.
8. The group G, Siegel, pp. 53–57.
9. Some simple games, Siegel, pp. 57–60.
10. Incentives and stops, Siegel, pp. 62, 68–80.
11. Canonical form, Siegel, pp. 64–67.
12. Special sums, Siegel, pp. 87–88.
13. Tiny and miny, Siegel, pp. 88–89.
14. Flowers, Siegel, pp. 91–93.

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How is the seminar organized? Do the following twice:

> Present a (chosen) “paper” during a 45 minutes lecture. Make slides, and use the blackboard.
>
> Produce a 7–10 page paper/report in LaTeX/PDF. Use your own words, no copy-paste; English.

Grading is based on the four Ps: presentation (2×), paper (2×), participation (including presence: discussions, questions) and maybe peer review OR programming.

Apply for participation: send e-mail† with proof of (*) from slide 5 before Friday afternoon February 8, 2019. At most ≈ 10 participants.

† w.a.kosters@liacs.leidenuniv.nl