Foundations of Computer Science

Fundamentele Informatica 1

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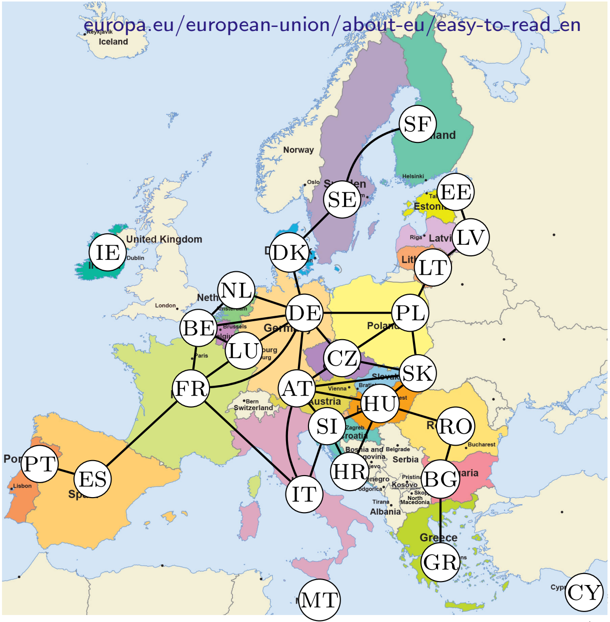
Universiteit Leiden

Najaar 2020

Universiteit Leiden
Leiden Institute of Advanced Computer Science

Hoofdstuk 4

Grafen


europa.eu/european-union/about-eu/easy-to-read_en
Iceland
Norway
United Kingdom
Dublin
London
Paris
Brussels
Bern
Switzerland
Vienna
Zagreb
Bosnia and
Belgrade
Serbia
Kosovo
Pristina
Tirana
Albania
North
Macedonia
Bucharest
Greece
Lisbon
Helsinki
Riga
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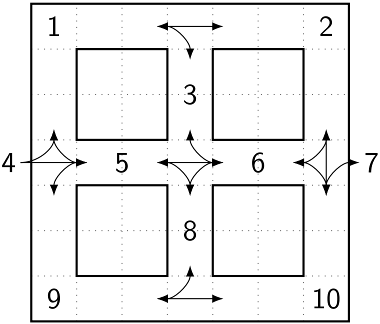
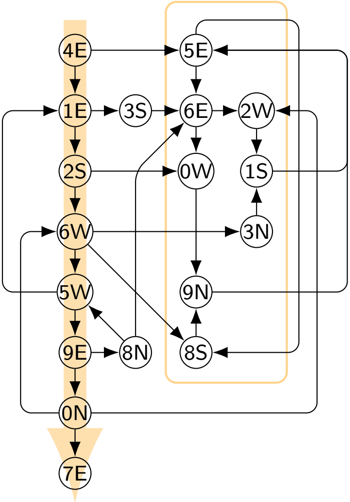
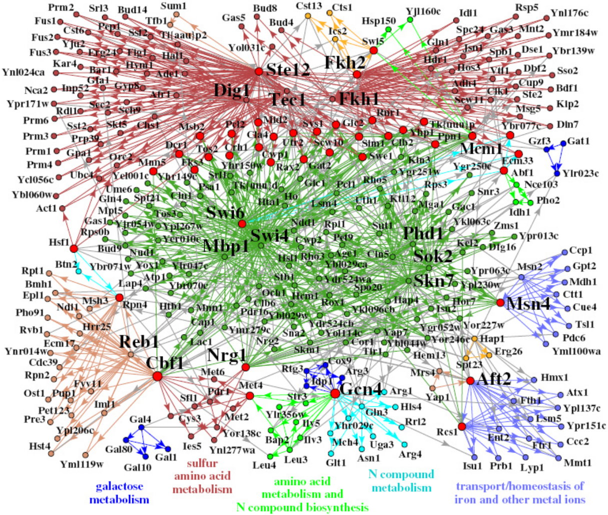




Robert Abbott multistate maze. mathpuzzle.com



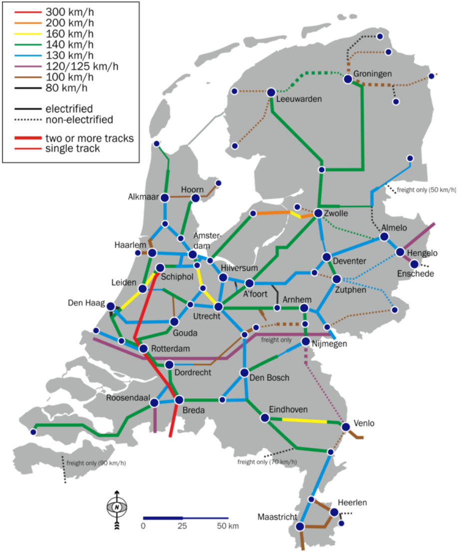
Directed network modules, Palla etal. New Journal of Physics, 2007.

zie ook college SNACS

‘families’ grafen

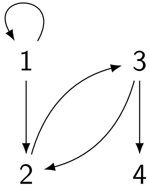
punten + verbindingen

- gericht vs ongericht
- parallelle verbindingen multi-
- info verbindingen gewogen
- identiteit knopen abstract



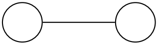
wikipedia spoorlijnen

$\{ (1,1), (1,2), (2,3), (3,2), (3,4) \}$

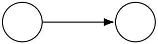


	1	2	3	4
1	1	1	0	0
2	0	0	1	0
3	0	1	0	1
4	0	0	0	0

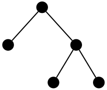
families



ONGERICHT
ch.8 Graph Theory



GERICHT
ch.9 Directed Graphs



ch.8.8 Tree graphs
ch.9.4 Rooted trees
ch.10 Binary Trees


KYKELIKY!

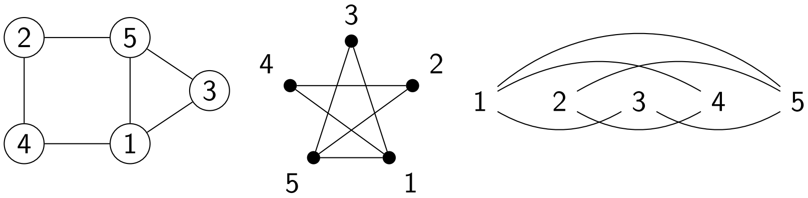
formalisatie

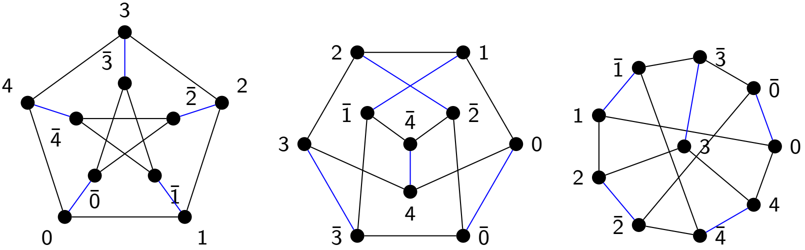
graaf $G = (V, E)$ verzamelingen V, E

- $V = V(G)$ knopen (punten; *vertices*, *nodes*)
- $E = E(G)$ lijnen (takken, zijden, kanten, bogen; *edges*, *arcs*)

lijn $\{u, v\}$ 'pair distinct vertices'

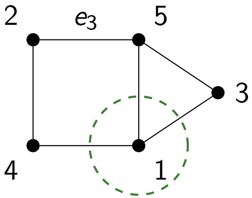
$V = \{ 1, 2, 3, 4, 5 \}$
$E = \{ \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 5\} \}$




3
$\bar{3}$
4
$\bar{2}$
2
$\bar{4}$
$\bar{0}$
$\bar{1}$
0
1
2
1
$\bar{1}$
$\bar{4}$
$\bar{2}$
3
0
4
$\bar{3}$
$\bar{0}$
$\bar{1}$
$\bar{3}$
$\bar{0}$
1
0
3
2
4
$\bar{2}$
$\bar{4}$

$G = (V, E)$ $e = \{u, v\}$ in E

- e *verbindt* u en v
- u *uiteinde* van e
- u en v *adjacent* (buren)
- u en e *incident*



graad van v aantal buren
$\deg(v)$
geïsoleerd $\deg(v) = 0$

$G = (V, E)$

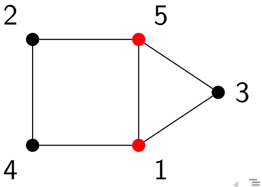
Thm. 8.1

som van de graden is twee keer aantal lijnen

$$\sum_{x \in V} \deg(x) = 2 \cdot |E|$$

gevolg

het aantal knopen met oneven graad is even



$G = (V, E)$ met $V = \{v_1, v_2, \ldots, v_n\}$ 'geordend'

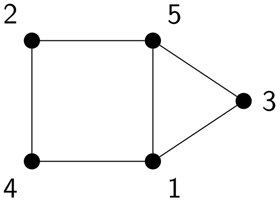
adjacency matrix burenmatrix

$n \times n$ matrix $A = (a_{ij})_{i,j=1\ldots n}$

$$a_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in E \\ 0 & \text{anders} \end{cases} .$$

ongerichte graaf:

- symmetrisch
- nullen op diagonaal

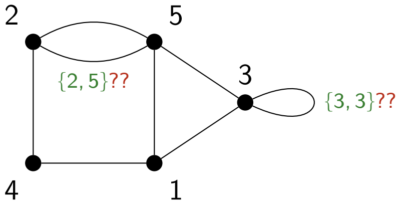


naar

van	1	2	3	4	5
1	0	0	1	1	1
2	0	0	0	1	1
3	1	0	0	0	1
4	1	1	0	0	0
5	1	1	1	0	0

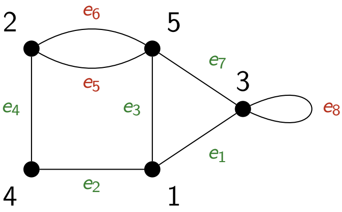
parallele lijnen

lus *loop*

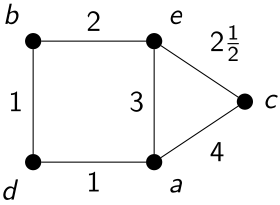


	1	2	3	4	5
1	0	0	1	1	1
2	0	0	0	1	2
3	1	0	1	0	1
4	1	1	0	0	0
5	1	2	1	0	0

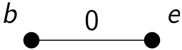
incidentie matrix

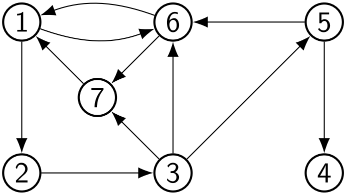


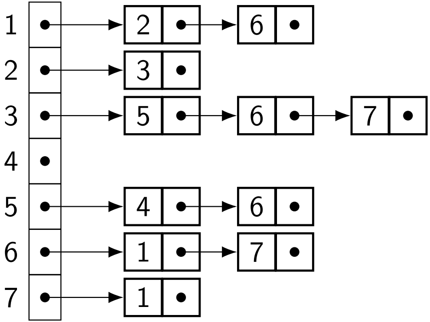
	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	1	1	1	0	0	0	0	0
2	0	0	0	1	1	1	0	0
3	1	0	0	0	0	0	1	1
4	0	1	0	1	0	0	0	0
5	0	0	1	0	1	1	1	0



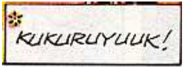
	a	b	c	d	e
a	0	0	4	1	3
b	0	0	0	1	2
c	4	0	0	0	$2\frac{1}{2}$
d	1	1	0	0	0
e	3	2	$2\frac{1}{2}$	0	0




1
6
5
7
2
3
4


1 → 2 → 6
2 → 3
3 → 5 → 6 → 7
4
5 → 4 → 6
6 → 1 → 7
7 → 1

Overzicht

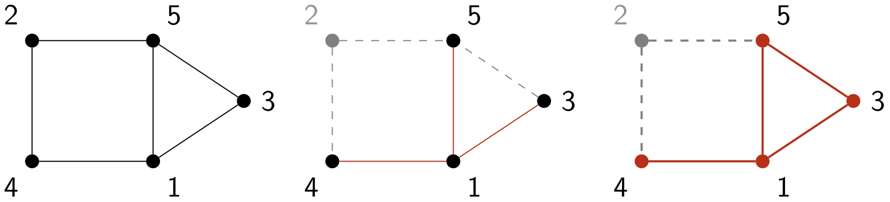


$G = (V, E)$

subgraaf $G' = (V', E')$ $\quad V' \subseteq V,\ E' \subseteq E$

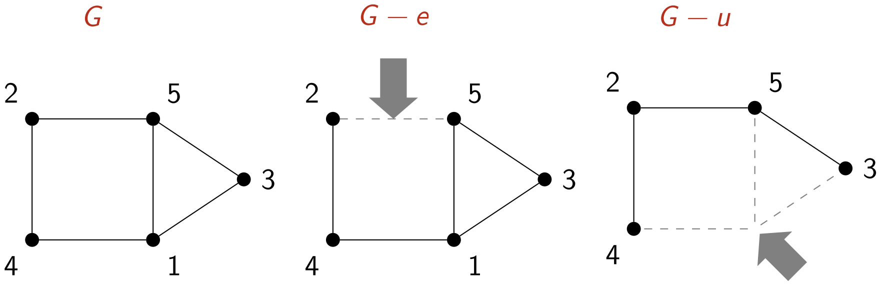
$V = \{1, 2, 3, 4, 5\}$ $\quad V' = \{1, 2, 3, 4, 5\}$

$E = \{ 13, 14, 15, 24, 25, 35 \}$ $\quad E' = \{ 13, 14, 15, 24, 25, 35 \}$ (luie notatie)

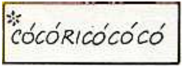


geïnduceerde subgraaf $G' = (V', E')$ $\quad V' = \{1, 2, 3, 4, 5\}$

$V' \subseteq V,\ E' = E \cap V' \times V'$ $\quad E' = \{ 13, 14, 15, 24, 25, 35 \}$


G
G − e
G − u
2
5
3
4
1
2
5
3
4
1
2
5
3
4

Overzicht



$v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_n, v_n \qquad e_k = \{v_{k-1}, v_k\}$

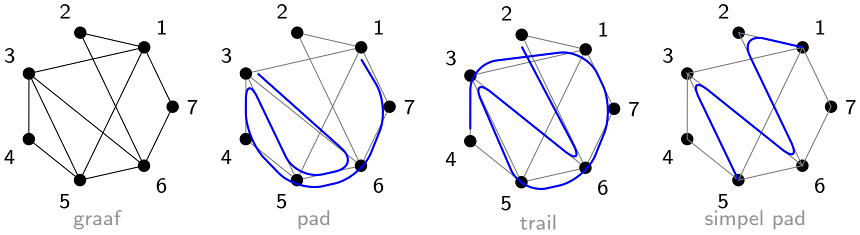
pad $\quad v_0, v_1, v_2, \ldots, v_n \quad \{v_i, v_{i+1}\} \in E \quad$ 1, 7, 6, 5, 4, 3, 5, 6, 3

van v_0 naar v_n tussen . . .

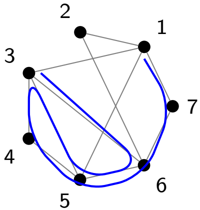
lengte n

trail verschillende *lijnen* 2, 6, 3, 5, 6, 7, 1, 3, 4

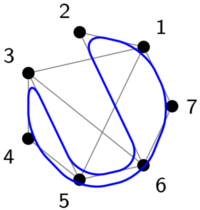
simpel pad verschillende *knopen* 1, 2, 6, 3, 5



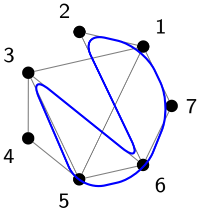
pad	$v_0, v_1, v_2, \ldots, v_n$	$\{v_i, v_{i+1}\} \in E$	1, 7, 6, 5, 4, 3, 5, 6, 3
gesloten pad	$v_0 = v_n$	kring	1, 2, 6, 5, 3, 4, 5, 6, 7, 1
circuit	verschillende lijnen		1, 2, 6, 3, 5, 6, 7, 1
cykel	verschillende knopen		1, 2, 6, 3, 5, 1



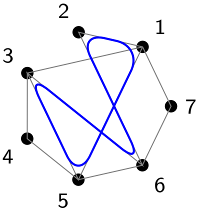
pad



gesloten pad



circuit



cykel

over Schaum

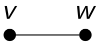
distinct		edge	vertex		edge	vertex
Schaum	path	trail	simple path	closed (path)	×circuit	≈cycle
Wiki	walk	trail	path	closed (walk)	circuit	cycle

Schaum “cycle (or circuit)” (zie p.160).

“Hamilton circuit” “Euler circuit”

heeft een kring een beginpunt?

$n \geqslant 3$



v, w, v géén cycle

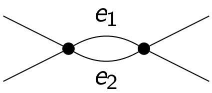
gewone (simpele) graaf

$$\text{simpel} \subseteq \text{trail} \subseteq \text{pad}$$

$$\text{verschillende knopen} \implies \text{verschillende lijnen}$$

multigraaf

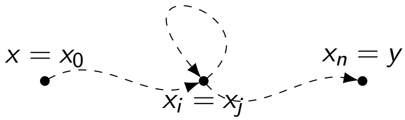
$$v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_n, v_n \qquad e_k \text{ tussen } v_{k-1} \text{ en } v_k$$



graaf G $\quad x, y \in V(G)$

Thm. 8.2.

Als er een pad is van x naar y in G, dan is er ook een *simpel* pad van x naar y.



$$x = x_0, x_1, \ldots, x_{i-1}, \boxed{x_i}, \overbrace{x_{i+1}, \ldots, x_{j-1}}, \boxed{x_j}, x_{j+1}, \ldots, x_n = y$$

$$x = x_0, x_1, \ldots, x_{i-1}, \boxed{x_i = x_j}, x_{j+1} \ldots, x_n = y$$

(herhalen)

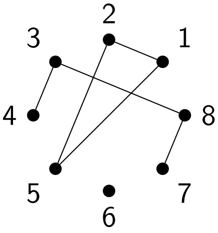
verbonden $x \sim y$ pad tussen x en y

equivalentierelatie

– $x \sim x$ reflexief pad van lengte nul

– als $x \sim y$ dan $y \sim x$ symmetrisch
pad omdraaien

– als $x \sim y$ en $y \sim z$ dan $x \sim z$ transitief
paden achter elkaar



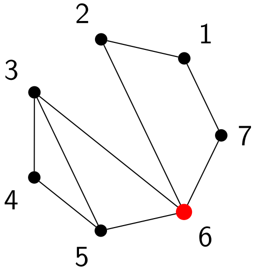
(samenhangs-)component

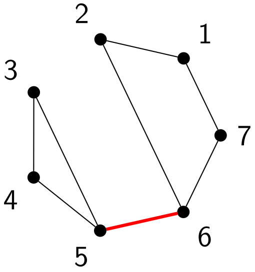
$\{1, 2, 5\}, \{3, 4, 7, 8\}, \{6\}$

aantal componenten neemt bij verwijderen toe

$G - v$ *articulatie punt* v (cutpoint)

$G - e$ *brug* e (cut edge)

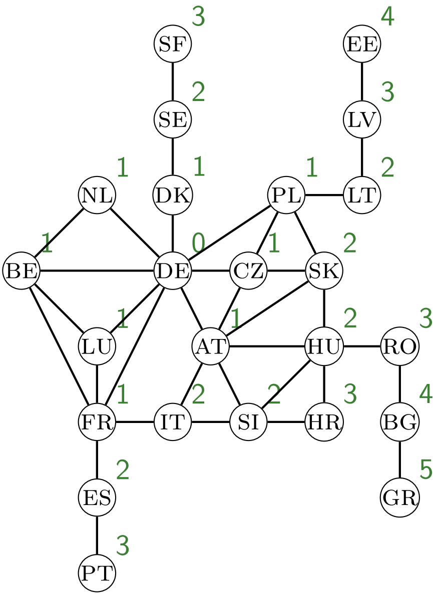




$d(x, y)$ afstand lengte kortste pad (gemeten in lijnen)

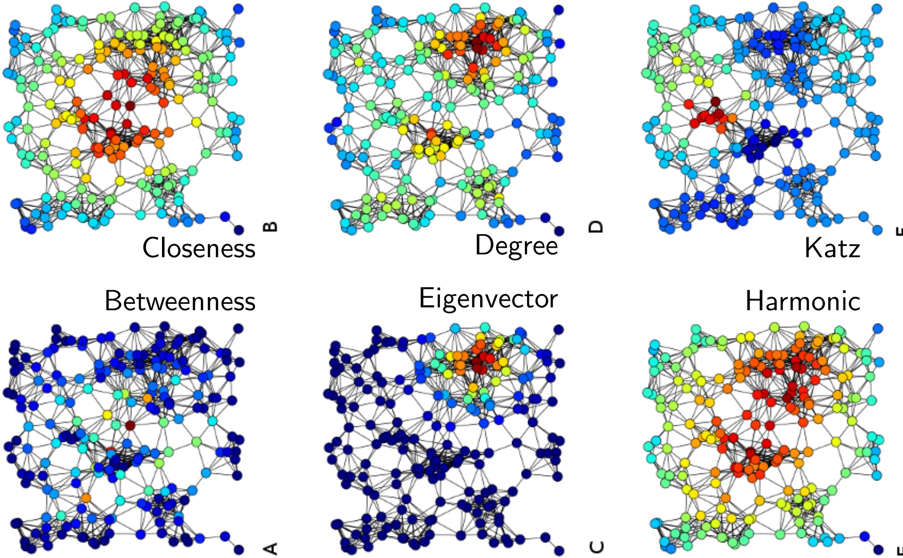
- $d(x, y) = 0$ desdals $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leqslant d(x, y) + d(y, z)$ *driehoeksongelijkheid*

diameter G langste afstand



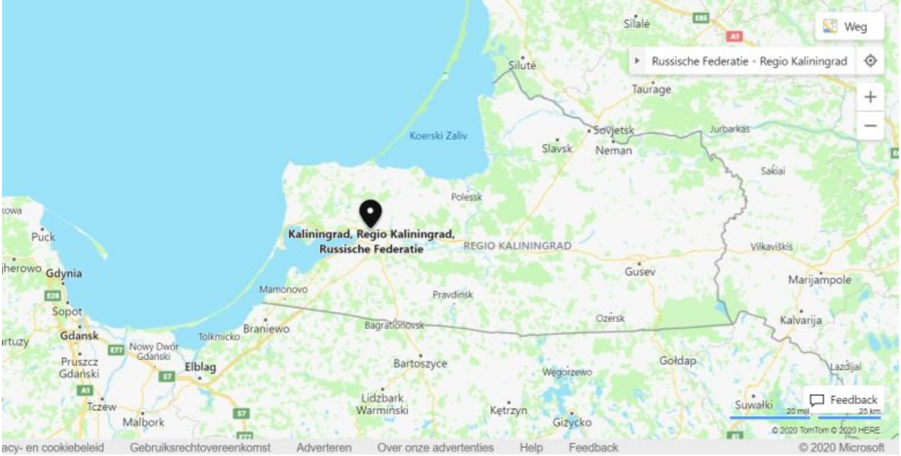
$d(\mathrm{GR}, \mathrm{EE}) \leqslant$
$d(\mathrm{GR}, \mathrm{DE}) + d(\mathrm{DE}, \mathrm{EE}) = 9$

wikipedia: Centrality




COCK-A-DOODLE-DOO

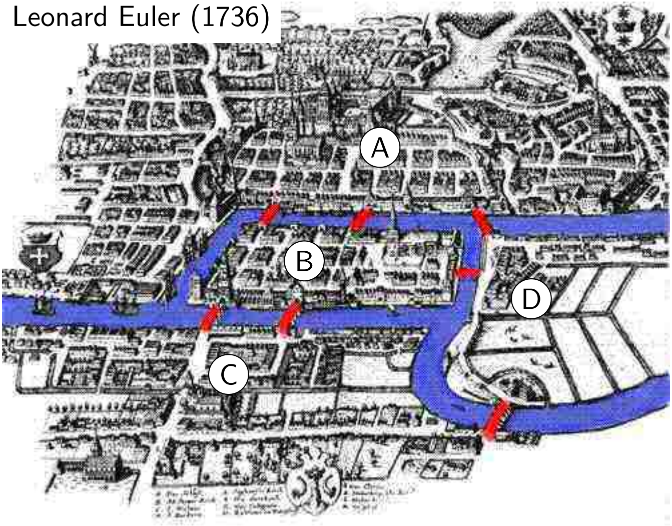
Kaliningrad / Königsberg in Preußen

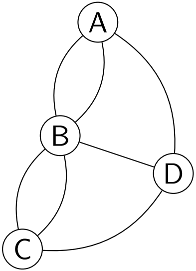


bing maps

Koningsberger bruggenprobleem

Leonard Euler (1736)





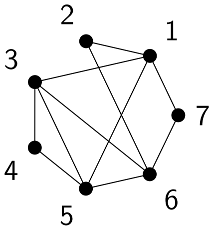
Leonhard Euler. Solutio problematis ad geometriam situs pertinentis

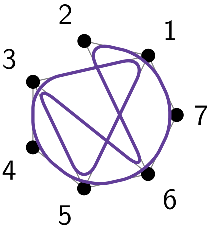
Euler circuit alle lijnen precies één keer

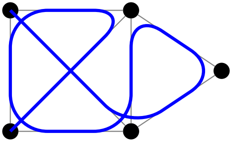
Euler graph heeft Euler circuit

Thm. 8.3

samenhangende graaf G is Euler desda elke knoop heeft even graad





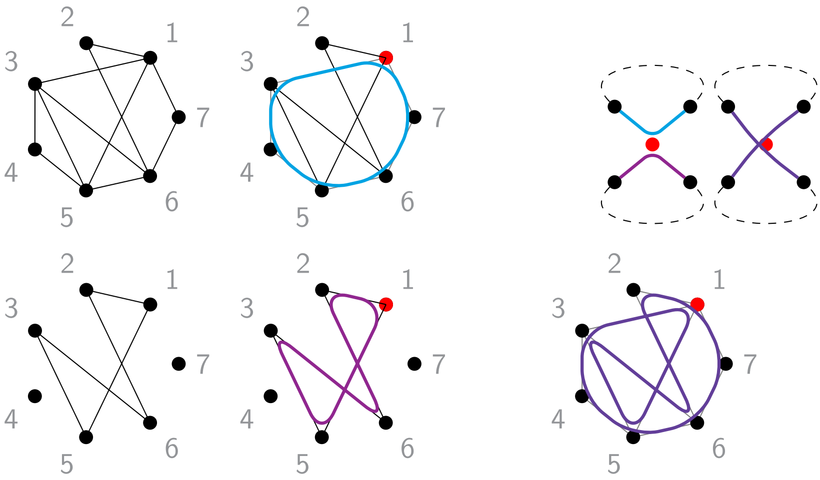


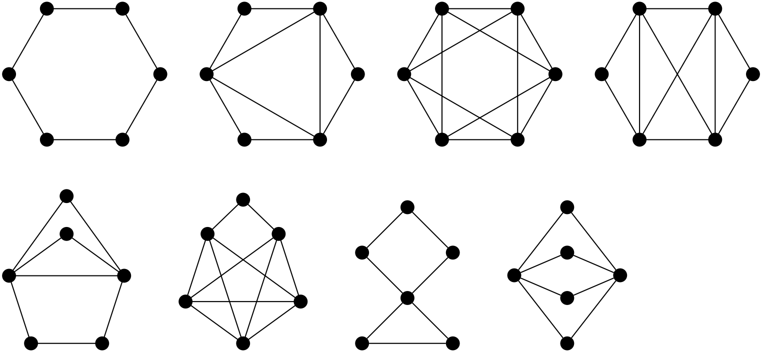
Cor. 8.4

Euler trail maximaal twee oneven graad

Euler circuit alle lijnen precies één keer

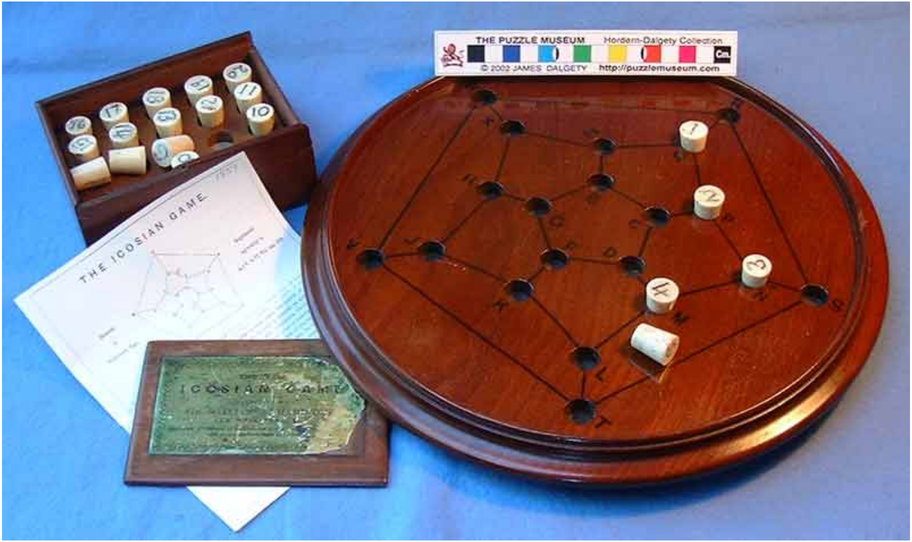
samenhangende graaf G is Euler desda elke knoop heeft even graad





☒'Sloane' A003049 online encyclopedia of integer sequences

icosian game

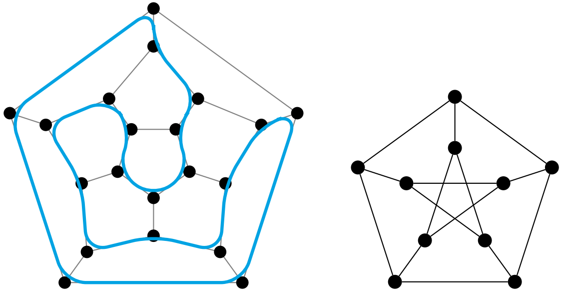


William Rowan Hamilton 1857

www.puzzlemuseum.com

Hamilton cykel   alle knopen precies één keer

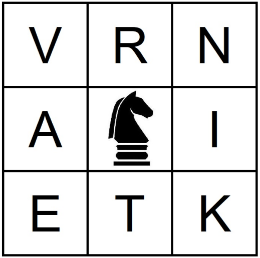
Hamilton graph   heeft Hamilton cykel



☒Ore's Theorem (1960)

A simple graph with n vertices ($n \geqslant 3$) is Hamiltonian if, for every pair of non-adjacent vertices, the sum of their degrees is n or greater.

$(u, v) \notin E$ dan $\deg(u) + \deg(v) \geqslant n$



wikipedia

	gen	die	ren	mernd	keln	a-	al-	spin-	
stun-	ih-	flim-	tra-	stern	häup-	nen	man-	dem	daß
ein	nacht	nen	sinkt	dun-	di-	zu	nacht	te	tel
die	den	die	hält	ter	kein	zen	von	geht	ter-
	ter-	die	rin-	es	en-	nun	jahr	es	
	zeit	wacht	de	heu-	stol-	eis	gen	mit-	
win-	den	son-	ei-	e	doch	nur	laut	in	zwölf-
nen-	die	der	te	grün	chen-	re	mal	ra-	dröhnt
durch	stur-	wald	wip-	ih-	treu-	lings-	wald	die	lich-
vom	wen-	weh'n	me	hen	hüllt	weiß	nen	me	und
im	und	na-	de	fel	früh-	tur-	feld	tes	tan-

Die Gartenlaube (1899) wikipedia

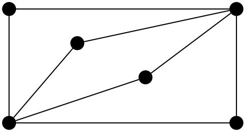
Hamilton cykel	alle knopen precies één keer
Hamilton graaf	heeft Hamilton cykel

Stelling

☒NP-compleet om te bepalen of G Hamilton is

Euler

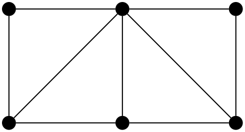
alle lijnen



eenvoudige karakterisatie

Hamilton

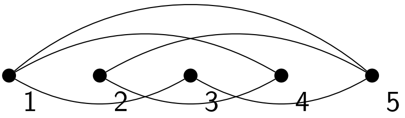
alle knopen



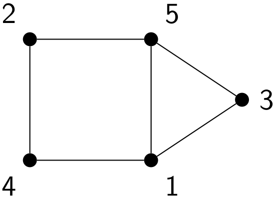
geen karakterisatie

computationally hard

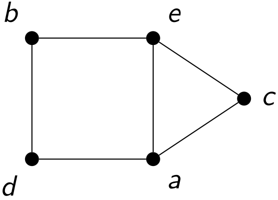

KUCKELIKUUU!



G

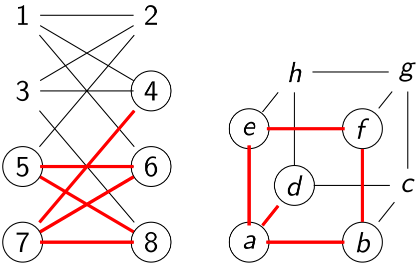


G'



u	1	2	3	4	5	6	7	8
$\varphi(u)$	h	g	c	d	f	e	a	b

behoudt aantal knopen, aantal lijnen, graden, paden, ...

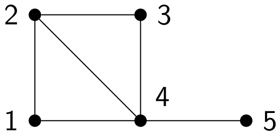


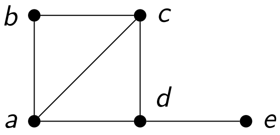
$G = (V, E) \qquad G' = (V', E')$

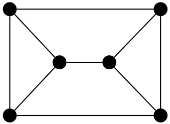
isomorfisme $\qquad \varphi : V \to V' \qquad$ bijectie

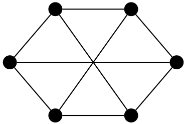
$(u, v) \in E$ desda $(\varphi(u), \varphi(v)) \in E'$

behoudt aantal knopen, aantal lijnen, graden, paden, . . .









‘abstracte’ grafen

Unsolved problem in computer science

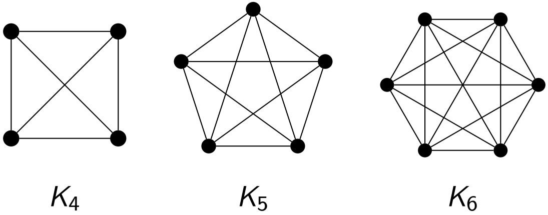
Can the graph isomorphism problem be solved in polynomial time?

P vs. NP

4 Grafen

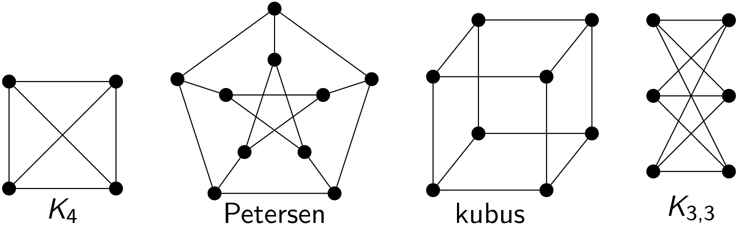
- Definities
- Deelgraaf
- Paden
- Euler en Hamilton
- Isomorfie
- **Speciale grafen**
- Vlakke grafen☒
- Gerichte grafen

K_n complete graaf



$$|V| = n \qquad |E| = \frac{n(n-1)}{2}$$

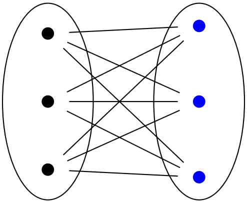
k-regulier alle knopen graad k



$|V| = n \quad |E| = \frac{k}{2}n$

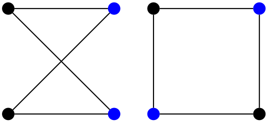
(compleet) bipartiet

$K_{m,n}$ *compleet bipartiet*

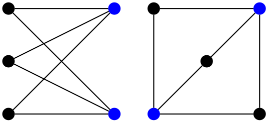


$$|V| = m + n$$
$$|E| = m \cdot n$$

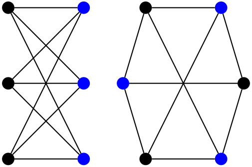
$K_{2,2}$



$K_{3,2}$

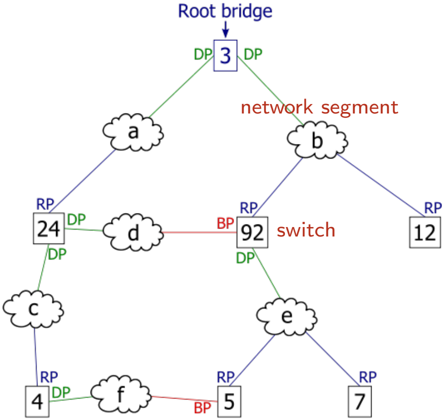


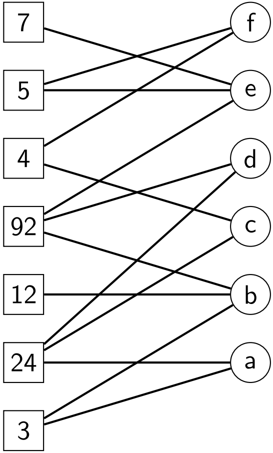
$K_{3,3}$



bipartiete graaf

lijnen alleen tussen twee (disjuncte) deelverzamelingen knopen





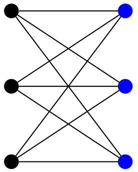
GhosT wikipedia

Thm. 8.11.

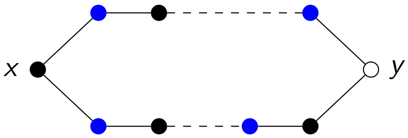
graaf G, equivalent zijn:

1. G is bipartiet
2. G heeft alleen cykels van even lengte
3. G is 2-kleurbaar

$(1,3) \implies (2)$

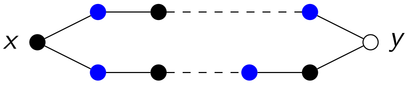


$(2) \implies (1,3)$



(1) $\Longrightarrow$ (2) Als de graaf bipartiet is loopt elk pad van een knoop naar zichzelf steeds heen en weer tussen de partities. Om terug te komen dus een even aantal stappen.

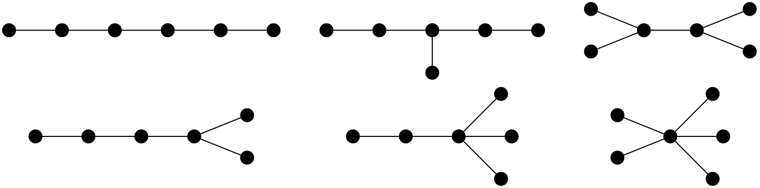
(2) $\Longrightarrow$ (1) Omgekeerd nemen we aan dat de graaf alleen kringen van even lengte heeft. We gaan de knopen uit de graaf zwart-blauw kleuren om zo de twee partities te onderscheiden. Kies een willekeurige knoop x van de graaf en kleur deze zwart. Als een knoop gekleurd is dan krijgen zijn buren de tegengestelde kleur. Dat leidt niet tot problemen: er is geen (ongekleurde) knoop y met zowel een zwarte als een blauwe buur. Dan zou namelijk y vanuit x te bereiken zijn met zowel een oneven als een even aantal lijnen, en vinden we een kring van oneven lengte.



boom

– samenhangend

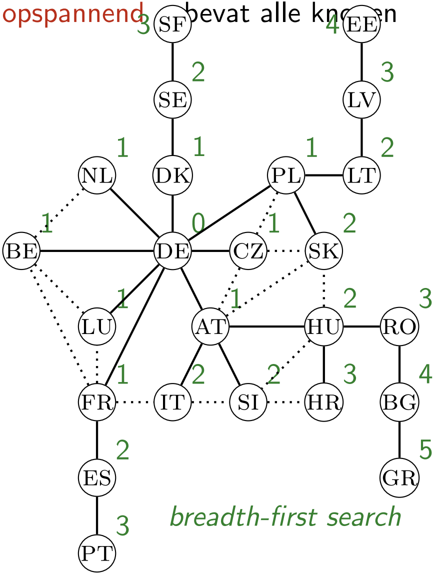
– acyclisch   geen cykels



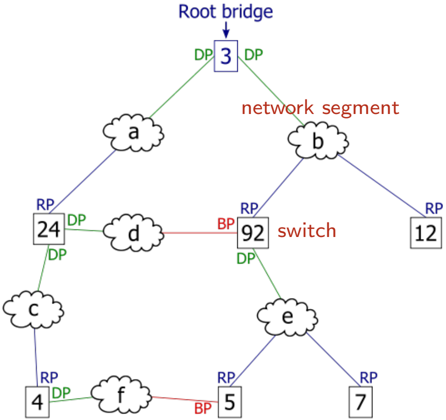
eigenschappen:

- tussen elk tweetal knopen precies één simpel pad
- $|E| = |V| - 1$

Sch 8.8 Tree graphs   Sch 10 Binary trees   (apart) opspannend bevat alle knopen

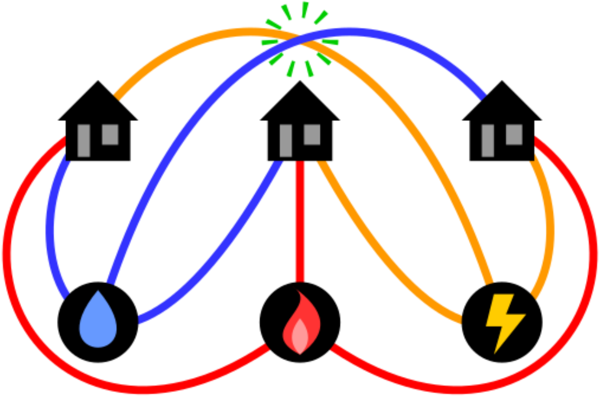


breadth-first search



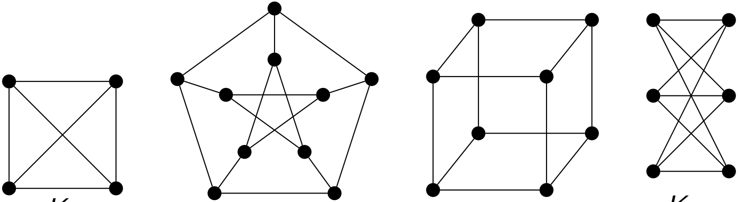

COCK-A-DOODLE-DO

vlakke graaf

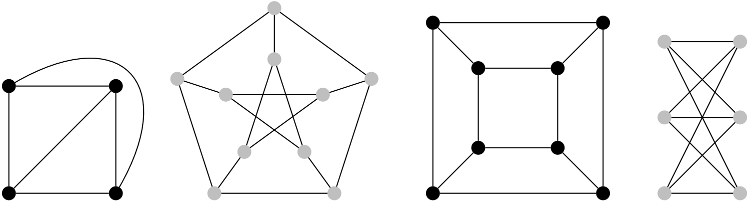


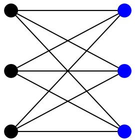
Cmglee wikipedia

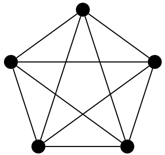
vlakke graaf zonder kruisende lijnen



K_4 Petersen kubus $K_{3,3}$



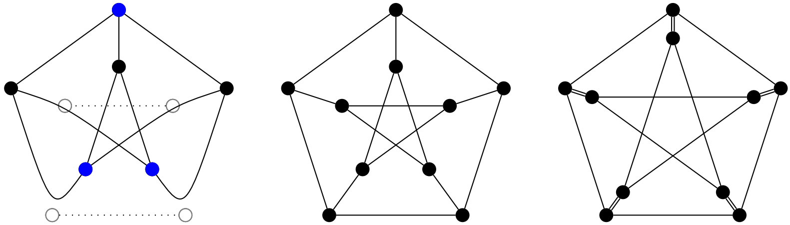




☒Thm. 8.10

Een graaf G is vlak desdals G niet $K_{3,3}$ of K_5 bevat

Kuratowski (1930) *Wagner* (1937)

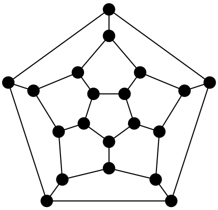
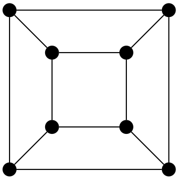
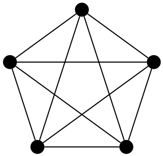
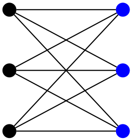


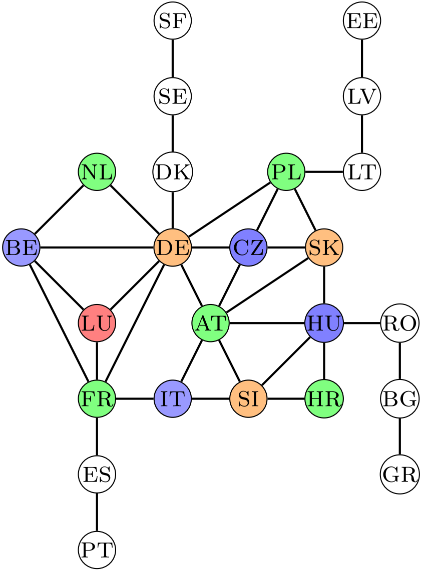
$G = (V, E)$ planaire graaf   vlakken R (*regions*)

⊠Thm. 8.8. Euler

$|V| - |E| + |R| = 2$

$\|V\| = 20,\ \|E\| = 30,$	$\|V\| = 8,\ \|E\| = 12,$	$\|V\| = 5,\ \|E\| = 10$	$\|V\| = 6,\ \|E\| = 9$
$\|R\| = 12$	$\|R\| = 6$	$\|R\| = 7?$	$\|R\| = 5?$
Thm.8.7. $5\|R\| = 2\|E\|$	$4\|R\| = 2\|E\|$	$3\|R\| \leqslant 2\|E\|$	$4\|R\| \leqslant 2\|E\|$



elke vlakke graaf heeft een 4-kleuring

Kempe 1879

Heawood 1890

☒**Thm.8.12** vijfkleurenstelling

Appel & Haken 1974

handwerk + computerbewijs

1,834 configurations


CUCURIGUUUU

gerichte graaf $G = (V, E)$ verzamelingen

- $V = V(G)$ knopen (punten; *vertices*, *nodes*)
- $E = E(G)$ pijlen (takken, kanten; *edges*, *arcs*)

pijl (u, v) geordend tweetal

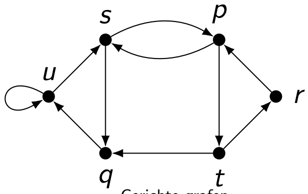
– e *van u naar v*

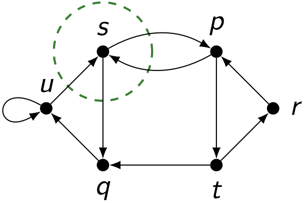
– u,v *begin*, *eind* van e

– $e = (u, u)$ *lus* *loop*

$V = \{ p, q, r, s, t, u \}$

$E = \{ (p,s), (p, t), (q, u), (r, p), (s,p), (s, q), (t, q), (t, r), (u, s), (u,u) \}$





naar

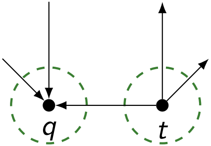
		p	q	r	s	t	u
van	p	0	0	0	1	1	0
	q	0	0	0	0	0	1
	r	1	0	0	0	0	0
	s	1	1	0	0	0	0
	t	0	1	1	0	0	0
	u	0	0	0	1	0	1

in/uit-graad van v

$indeg(v)$ $outdeg(v)$

bron $indeg(v) = 0$ source

put $outdeg(v) = 0$ sink

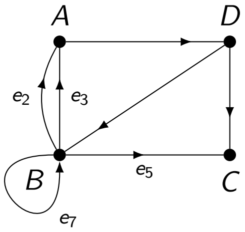


Thm. 9.1

$$\sum_{x \in V} indeg(x) = \sum_{x \in V} outdeg(x) = |E|$$

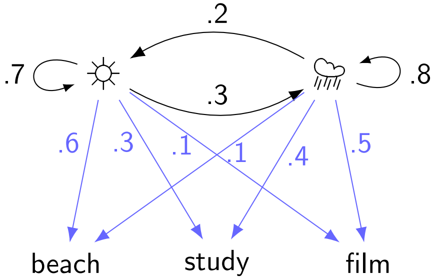
$\{\ldots, (B, A), (B, A), \ldots\}$

(gerichte) multigraaf



graaf met gewichten

☒Hidden Markov Model



$v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_n, v_n$

pad $v_0, v_1, v_2, \ldots, v_n$ $(v_i, v_{i+1}) \in E$

van v_0 naar v_n

lengte n aantal pijlen

– trail verschillende *lijnen*

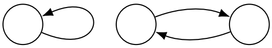
– simpel pad verschillende *knopen*

gesloten pad $v_0 = v_n$ kring

– circuit verschillende lijnen

– cykel verschillende knopen

opspannend pad bevat alle knopen



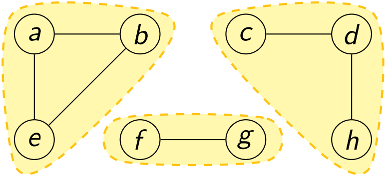
graaf sterk samenhangend   voor elke x en y is er een pad van x naar y

ongericht

pad tussen x en y

connected component

samenhangs-component



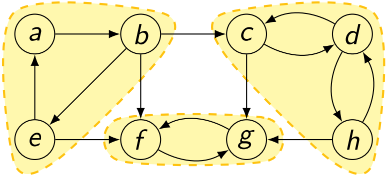
gericht

pad van x naar y

én pad van y naar x

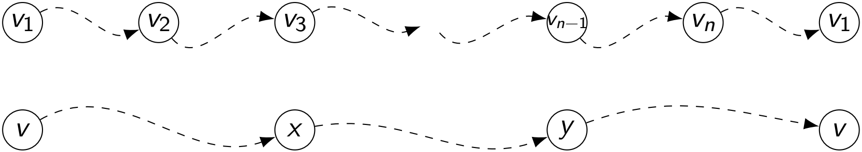
strongly connected component

sterke samenhangs-component



Thm. 9.2

gerichte graaf G
sterk samenhangend desda heeft een gesloten opspannend pad

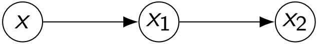


ongerichte graaf G
samenhangend desda heeft een opspannend pad

Thm. 9.2

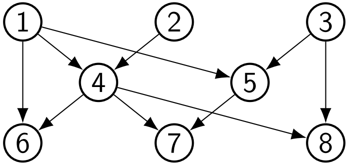
gerichte graaf G

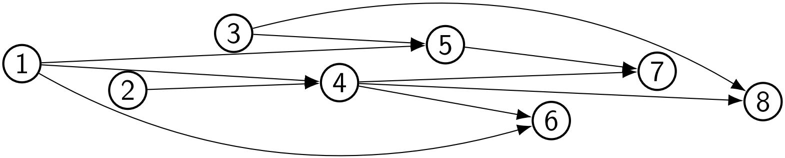
zonder cykels   dan heeft G een put en een bron



$G = (V, E)$ gericht

topologische ordening van G $\quad (v_1, \ldots, v_n) \quad (v_i, v_j) \in E$ dan $i < j$





$(1, 2, \ldots, 7, 8) \quad (2, 1, 4, 6, 3, 5, 8, 7)$

Thm. 9.8

gerichte graaf G zonder cykels $\quad$ dan bestaat er een topologische ordening

END.

edit 7 december 2020